ELEMENTARY LINEAR ALGEBRA
Problem List 1
Mathematical induction, binomial formula, complex numbers

1. Apply mathematical induction to show that the following equations hold for all $n \in \mathbb{N}$:

   (a) $1 + 3 + \cdots + (2n - 1) = n^2$,

   (b) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$,

   (c) $1 + 3 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}$.

2. Apply mathematical induction to shown that the following inequalities hold:

   (a) $2^n > n^2$ for $n \geq 5$,

   (b) $n! > 2^n$ for $n \geq 4$,

   (c) $(1 + x)^n \geq 1 + nx$ for $x \geq 0$ and $n \in \mathbb{N}$.

3. Using Newton's binomial formula, expand the following powers:

$$(a)\ (x - 2y)^4, \quad (b)\ (a + \sqrt{2})^5, \quad (c)\ (c + \frac{1}{c^2})^5.$$

4. (a) Find the coefficient standing by x^5 in the expansion of $(x^3 + \frac{1}{x^2})^{15}$
   (b) Find the coefficient standing by $\sqrt[4]{y}$ in the expansion of $(\sqrt[4]{y^5} - \frac{2}{y^3})^7$.

5. Perform the algebraic operations and write the result in the form $a + ib$:

$$(a)\ (1 + i)(2 - 3i), \quad (b)\ (-6 + 5i) + (2 - 4i), \quad (c)\ (-5 + \sqrt{2}i) - (2 - i)$$

$$(d)\ (1 + i)(2 - i)(3 + 2i), \quad (e)\ (1 - 2i)^3, \quad (f)\ (1 + i)^4, \quad (g)\ (-2i)^6,$$

$$(h)\ \frac{1 + 2i}{2 - 3i}, \quad (i)\ \frac{2 - \sqrt{2}i}{2 + \sqrt{3}i}, \quad (j)\ \frac{1 + 3i}{3 + 4i} + \frac{1 - 4i}{3 - 4i}, \quad (k)\ 2 - 3i + \frac{1 - 2i}{i + 2}$$

6. We define the n-th power of the complex number z in the natural way, namely

$$z^0 = 1, \quad z^n = z^{n-1} \cdot z, \quad z^{-n} = 1/z^n$$

   for $n \geq 1$. Compute i^n for $n \in \mathbb{Z}$ and $(1 + i)^n$ for $n = 1, 2, 3, 4$.

7. Using mathematical induction, show that the following formula holds:

$$1 + z + z^2 + \ldots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

   for $z \in \mathbb{C} \setminus \{1\}$.

8. Comparing the real and imaginary parts of both sides of the equations, solve them for real x, y:

$$(a)\ (1+i)x + (1-2i)y = 1-i, \quad (b)\ \frac{x-3}{1+i} + \frac{y+3}{1-i} = 1+i,$$

$$(c)\ x^2 + iy^2 = 1+2i, \quad (d)\ x^2 - iy^2 = 1+i.$$

9. Writing z in the form $z = x+iy$, solve the following equations:

$$(a)\ z^2 = i, \quad (b)\ z^2 = -i, \quad (c)\ 4+2i = (1+i)z,$$

$$(d)\ z^2 + 4i = 0, \quad (e)\ \frac{z+2}{i-1} = \frac{3z+i}{2+i}, \quad (f)\ z^2 - 6z + 10 = 0.$$

$$(g)\ 2z + (3-i)\overline{z} = 5+4i, \quad (h)\ z+i = \overline{z+i}, \quad (i)\ z\overline{z} + (z-\overline{z}) = 3+2i,$$

$$(j)\ z + \overline{z} + i(z-\overline{z}) = 5+3i, \quad (k)\ i\mathrm{Re}z + i\mathrm{Im}z = 2i-3, \quad (l)\ \overline{z} = z^2.$$

Indicate the solution on the complex plane.

10. Find all complex numbers z which satisfy the following conditions:

$$(a)\ \mathrm{Re}z - 3\mathrm{Im}z = 2, \quad (b)\ \mathrm{Re}(iz) \geq 1, \quad (c)\ \mathrm{Im}(iz) \leq 2.$$

Indicate the solution on the complex plane.

11. Compute the modulus of each of the following complex numbers:

$$2+7i, \quad \frac{4+i}{3+2i}, \quad (1+\sqrt{2}i)^4, \quad \frac{(3-\sqrt{3}i)^2}{(\sqrt{2}+2i)^3}$$

12. Write the following numbers in the trigonometric form:

$$(a)\ -3i, \quad (b)\ 1+\sqrt{3}i, \quad (c)\ 2-2\sqrt{3}i, \quad (d)\ \left(\frac{\sqrt{3}-i}{1+i}\right)^3.$$

13. Using de Moivre's formula, compute the following powers:

$$(a)\ (1+i)^{11}, \quad (b)\ (2-2\sqrt{3}i)^7, \quad (c)\ \left(\frac{1-i\sqrt{3}}{1-i}\right)^{10}.$$

14. Draw on the complex plane the sets of complex numbers satisfying the following conditions:

$$(a)\ |z+i| = 5, \quad (b)\ |z-1| < 3, \quad (c)\ 1 \leq |z+i| \leq 2, \quad (d)\ |z-i| = |z+i|.$$

$$(e)\ \mathrm{Im}(z^3) < 0, \quad (f)\ \mathrm{Re}(z^4) \geq 0, \quad (g)\ \mathrm{Im}(z^2) \geq \mathrm{Re}((\overline{z})^2).$$

15. Using the algebraic form of complex numbers, compute the following roots:

$$\sqrt{2-i}, \quad \sqrt{3-2i}, \quad \sqrt{1+i2\sqrt{3}}.$$

16. Using the trigonometric form of complex numbers, compute the following roots:

$$\sqrt[6]{1}, \quad \sqrt[3]{2+i}, \quad \sqrt[4]{-16}$$

17. Solve the equations for complex z:

$$(a)\ z^2+z+1=0, \quad (b)\ z^2+9=0, \quad (c)\ z^4-2z^2+4=0, \quad (d)\ z^2+(1+i)z-i=0,$$

$$(e)\ z^4=1, \quad (f)\ z^2+3iz+4=0, \quad (g)\ z^3=(1-i)^3, \quad (h)\ (z-i)^4=(iz+4)^4,$$

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(the problems are taken from the book of Gewert and Skoczylas, with some added by myself)